Heston
Stochastic Volatility
Model of Stock Prices

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Master of Science in Financial and Industrial Mathematics, is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed

Student No 58211432

26th August 2009
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Abstract

There are problems with the usual assumption of a log normal distribution of stock prices, the most important discrepancy is to be found in the tails and in time dependent volatility. To get a more accurate model of the distribution of stock prices one suggestion is to use a stochastic volatility (SV) model. This project concentrates on Heston’s stochastic volatility model. This provides a very useful way to price exotic options based on the prices for plain vanillas. Heston’s model is particularly useful as it is possible to find explicit formulas for the distribution of prices for Heston’s SV model. Further improvements are mentioned namely adding jumps to the stochastic volatility model, (SVJ) and the SABR model.
1 Introduction

1.1 Basic Introduction to Options

Consider the following example. We do not know how much oil will cost in 12 months, and yet many businesses need to plan ahead and sign contracts, this requires some assumptions about the unknown price in a year’s time. One way to avoid nasty surprises is to agree a price now which will be paid in 12 months when the oil is delivered. This is called a forward contract. In a forward contract no money changes hands at the start and the two parties agree the price of an item to be delivered at some specified date in the future.

There is still a problem. We know how foolish we would feel if it turned out that we had agreed to pay a higher price than we could have paid on the open market. Wouldn’t it be good to have the means to know in advance whether to take the forward price or the market price? This is the reason options are so useful, they allow us to select a price in advance, known as the strike price, and make the choice whether to trade at this price when we find out the market price on the expiry date. Such a choice isn’t free. One party usually pays the other some money initially in order to enter the contract. How much should such an option cost? Naturally in the absence of a time machine we do not know the future price of oil or of any other risky asset. We can however make intelligent guesses as to the likely outcomes. That is, we can postulate models of the probability distribution and use these to price the options. We concern ourselves only with options which are only able to be exercised on their expiry date, these are known as European options. We do not deal with American options which can be exercised at times before their expiry date.
Section 2 deals with the Black Scholes Merton model of asset prices which assumes constant volatility (CV). This model is compared with empirical results. An improvement to this model is then proposed in Section 3, the Stochastic Volatility model (SV), and some general properties of this are found. The Heston model is a particular version of the SV model and it is examined in some detail in Section 4. Some other models notably the Jump Diffusion model (SVJ) and SABR models are mentioned in Section 5. There is a short conclusion in Section 6 and the appendices contain the Matlab and Maple code used in the calculations.

1.2 Itô’s Lemma and Stochastic Processes

Use is made of Itô’s Lemma in this report, we begin with a brief description of a Stochastic Process. Let $(\Omega, F, T)$ be a probability space. A stochastic process is a set of random variables $\{S_t | t \in [0, T]\}$ defined on $(\Omega, F, T)$.

For our purposes the random variables will take values on the Real numbers, and in many cases the positive Reals. We cannot predict future values of a random variable, $S_t$, but if we know the distribution of the stochastic process we may calculate the probability of various outcomes. There are equations which try to model the rates of change of deterministic and stochastic variables. Itô calculus provides the extension of regular calculus to stochastic variables so that we may make use of such ideas. The basic statement of Itô’s Lemma for $f \in C^{1,2}$ and a stochastic process $S$, is

$$df = \frac{\delta f}{\delta t} dt + \frac{\delta f}{\delta S} dS + \frac{1}{2} \frac{\delta^2 f}{\delta S^2} (dS)^2. \quad (1.1)$$

The multiplication rules for $W_1$ and $W_2$, two standard Brownian motions with
correlation $\rho$, are shown in Table 1.1 on page 8.

<table>
<thead>
<tr>
<th>Product</th>
<th>$dt$</th>
<th>$dW_1$</th>
<th>$dW_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dt$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$dW_1$</td>
<td>0</td>
<td>$dt\rho$</td>
<td>$\rho dt$</td>
</tr>
<tr>
<td>$dW_2$</td>
<td>0</td>
<td>$\rho dt$</td>
<td>$dt$</td>
</tr>
</tbody>
</table>

Table 1.1: Multiplication table for Itô products.

The Lemma allows us to understand the interaction of stochastic and normal calculus. That is, it allows us to define an integral over a stochastic variable which is compatible with our definitions for deterministic variables.

It is more formally expressed in integral form:

$$
\int df = \int \frac{\delta f}{\delta t} dt + \int \frac{\delta f}{\delta S} dS + \int \frac{1}{2} \frac{\delta^2 f}{\delta S^2} (dS)^2.
$$

If $f$ depends on two stochastic processes, $X$ and $Y$, we use the following:

$$
df = \frac{\delta f}{\delta t} dt + \frac{\delta f}{\delta X} dX + \frac{1}{2} \frac{\delta^2 f}{\delta X^2} (dX)^2 + \frac{\delta f}{\delta Y} dY + \frac{1}{2} \frac{\delta^2 f}{\delta Y^2} (dY)^2 + \frac{1}{2} \frac{\delta^2 f}{\delta X \delta Y}. \quad (1.2)
$$

This is applied in (3.6).

### 1.3 Put Call Parity for European Options

A call option is the right to buy the underlying asset for the strike price, $K$ at the expiry time, $T$. A put option is the right to sell the underlying asset for the strike price $K$ at the expiry time $T$. There is of course, a relationship between the value of the call and put option, which is known as put call parity. It is common practice to evaluate call options and use put call parity to calculate put option prices from
the call option prices. We will do this in the explicit solution calculations in table 2.1.

A portfolio consisting of a call option and cash equal to the present value of \( K \), has the same value at expiration as a portfolio of a put option and the underlying asset. To see this, consider the first portfolio, a call option and cash which will be worth \( K \) at expiry; this portfolio will be worth the maximum of \( K \) or the stock it can purchase. At expiry the second portfolio is worth the maximum of the stock price or \( K \), because you have the right to sell the stock for \( K \). Since both are worth the same at expiry and we assume there is no arbitrage, (risk free profit), then they are worth the same at the start. Let

- \( K \), be the strike price,
- \( S \), be the present value of the risky asset,
- \( T \), be the time gap from now to expiry,
- \( r \), be the risk free rate of interest with continuous compounding,
- \( P \), be the value of a put option and
- \( C \), be the value of a call option, then the following holds:

\[
C + Ke^{-rT} = P + S. \tag{1.3}
\]
2 Constant Volatility Model (CV)

The classical model of stock prices is known as the Black Scholes Merton model, it was published in 1973. It assumes that the volatility is constant. Thus the dynamics of the prices of risky assets can be modelled by geometric Brownian motions (2.1). This leads to a distribution which is lognormal. The following notation will be used:

- $S_t$ is the value of the risky asset at time $t$, (stock, commodity or other security),
- $r$ is the risk free interest rate,
- $W_t$ is a standard Brownian motion with zero mean and variance $t$ and,
- $\sigma$ is the constant volatility. Then the price dynamics is given by

$$dS_t = rS_t dt + \sigma S_t dW_t.$$  \hfill (2.1)

It is possible to calculate prices of derivatives within this model. A derivative is a financial contract whose value is a function of the value of an underlying asset. In particular we are interested in options but there are other kinds of derivatives. In order to derive the price of a derivative, let us consider a portfolio which has a value of $\Pi$, and is composed of of minus one derivative, and $\frac{\delta f}{\delta S}$ units of risky asset, where $f(S)$ is the value of the derivative. Then we have,

$$\Pi = -f + \frac{\delta f}{\delta S} S_t.$$ 

We now consider the change in value of the portfolio, $d\Pi$, during the time from
\[ t \text{ to } t + dt, \]
\[ d\Pi = -df + \frac{\delta f}{\delta S} dS \]
\[ = -\frac{\delta f}{\delta t} dt - \frac{\delta f}{\delta S} dS - \frac{1}{2} \sigma^2 S_t^2 \frac{\delta^2 f}{\delta S^2} dt + \frac{\delta f}{\delta S} dS \]

where the second equation is obtained using Itô’s Lemma (1.1).

We see that the right hand side of this equation is risk free as it has only \( dt \) terms, therefore we expect the risk free rate of return from this portfolio. Hence,

\[ d\Pi = r\Pi dt = r \left[ -f + \frac{\delta f}{\delta S} S_t \right] dt. \]

Comparing the previous equations we derive the Partial Differential Equation (PDE),

\[ -\frac{\delta f}{\delta t} - \frac{1}{2} \sigma^2 S_t^2 \frac{\delta^2 f}{\delta S^2} = -rf + rS_t \frac{\delta f}{\delta S}, \]

with the boundary condition,

\[ f(S) = \max\{S - K, 0\}. \]

### 2.1 Black Scholes Merton (BSM) Formula for Option Prices

Rearranging terms of (2.2) gives us the much celebrated Black Scholes Merton PDE,

\[ \frac{\delta f}{\delta t} + rS_t \frac{\delta f}{\delta S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\delta^2 f}{\delta S^2} = rf, \]
with the boundary condition,

\[ f(S) = \max \{ S - K, 0 \} . \]

This allows one to calculate the fair value price of call and put options in the CV model. Consider a call option whose value \( f(S) \) is given by, \( f(S) = \max(S_t - K, 0) \). Then the solution to the PDE is given by the Black Scholes Merton Formulae for the value of a European plain vanilla call option \( C_0 \) and put option \( P_0 \) at time \( t = 0 \). Let

- \( K \) be the strike price,
- \( T \) be the time to expiry,
- \( S_0 \) be the initial value of the stock,
- \( r \) be the risk free rate of return and
- \( G \) be the Gaussian cumulative distribution. Then the BSM equations are,

\[
C_0 = S_0 G(d_1) - Ke^{-rT} G(d_2),
\]

and

\[
P_0 = Ke^{-rT} G(-d_2) - S_0 G(-d_1),
\]

where

\[
d_1 = \frac{1}{\sigma \sqrt{T}} \left( \log \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T \right), \quad d_2 = d_1 - \sigma \sqrt{T}.
\]

Finding the price of an option reduces to making an estimate of only one parameter,
the volatility $\sigma$. This is because we have been given the terms of the contract which will contain $K$ and $T$, and knowing $S_0$ from the present market valuations and $r$ the risk free rate, we may calculate the price of the option if we know $\sigma$, the volatility. Conversely we may calculate the volatility knowing the price of the option. This calculated volatility will be known as the implied volatility, $\sigma_{imp}$. In the CV model the volatility of a risky asset should be a property of the asset, not a property of the derivative. That is, the time to expiry or the strike price should not affect the value of the implied volatility, $\sigma_{imp}$. Therefore if we make a graph of the implied volatility against $T$ or $K$ we should have a straight line. This is not the case.

2.2 Comparison with Empirical Data

2.2.1 Histogram

The constant volatility model is not verified by financial data. Fig 2.1 uses the closing prices of the FTSE All Share index from 30th June 1995 to 4th June 2009. The histogram and normal curve have been calculated using SPSS 14.0. Fig 2.1 is typical of histograms of index values and stock values in that it shows too high a frequency of small changes, and too little of moderate changes. What is of more concern is that the extreme changes are more frequent than one would expect for a lognormal distribution. This is of great interest as the extreme changes are very significant (Taleb, 2007, Fig 14, p276). For a comparison with the daily returns of the S&P 500 index see Gatheral, (Gatheral, 2006, p3) and for a comparison with 30 minute returns of the S&P 500 see Cont and Tankov (Cont and Tankov, 2004, p212).
Fig. 2.1: Histogram showing the log of the daily change in FTSE All Share values from 1995 to 2009. It is evident that the data are not normally distributed.

2.2.2 Quantile Plot

The Quantile Plot fig 2.2 provides strong evidence to reject the hypothesis that the FTSE All Share prices have a lognormal distribution. If they were lognormal then we would have a straight line instead of an 'S' shape. The shape indicates that there are more data on the tails than would be expected in a lognormal distribution. This plot was generated using Matlab 7.7.0 (R2008b).
2.2.3 Comparison With Short Expiry Data

We calculate the price of calls and puts using the BSM formulae and the data from 15th September 2005 given by Gatheral (Gatheral, 2006, p40). The calculations use $\bar{v} = 0.0354$ for the constant volatility, $\sigma$ of the BSM formula, and time of $T = \frac{1}{250}$ since they expired the next day. The calculations use a risk free rate of return of 3.636 It is seen that the BSM formula is reasonably successful for in the money calls and puts getting $\frac{11}{12}$ of the calls and $\frac{4}{5}$ of the puts inside the bid ask spread. However the BSM gets all of the out of the money options below their bid
Table 2.1: Comparison of Black Scholes Merton Option prices and actual prices on 15th September 2005 for
SPX options expiring on the next day. The BSM figures are good for in the money options but are not inside the
bid ask spread for out of the money options. Note that the BSM formula used \( \bar{v} \) from the Heston parameters.

2.2.4 The Volatility Surface

It has become standard practice for the price of an option to be quoted not in units of currency, but by quoting the implied volatility \( \sigma_{imp} \). This is useful in that it allows one to compare options for different risky assets. If market prices for options are used to calculate the implied volatility \( \sigma_{imp} \) we discover that \( \sigma_{imp} \) is not constant, it shows a curve called a smile. If we plot against strike price and expiry time we get what is known as the volatility surface. This surface is not flat.
2.2.5 Conclusion of Comparison with Empirical Data

There are clear weaknesses with the CV model, particularly with its prediction of the frequency of extreme events. The CV model would imply that large movements of stock prices are very rare; however these movements have been observed much more frequently than the CV model suggests. This is seen in the histogram Fig 2.1 and in the quantile plot Fig 2.2. The comparison with short expiry data for 15th September 2005 in table 2.1 shows that the BSM model was able to give prices for in the money options within the bid ask spread, but was below the bid ask spread for all out of the money options. Finally, and most significantly the volatility surface is not flat, which the CV model assumes. Thus we need to change the model.
3 Stochastic Volatility (SV)

The improvement we consider for the Constant Volatility model is that we allow the volatility to vary in a stochastic manner. The Heston model in the next chapter is a specific example of this idea. We now treat volatility as a random variable. Dupire’s equation for local volatility, $\sigma$ will be considered in a subsection. Firstly we examine some properties of the stochastic volatility model. The dynamics of a stochastic volatility model (SV) may be depicted as follows. Let

- $S_t$ be the stock price at time $t$,
- $\mu_t$ be the drift at time $t$,
- $v_t$ be the instantaneous variance at time $t$ and
- $\eta$ be the variance of $v_t$.

$Z_1$ and $Z_2$ are a standard Brownian motions, with correlation of $\rho$, $\theta$ and $\omega$ are functions of $S_t, v_t$ and $t$. Then we have

\[dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dZ_1,\]  
(3.1)

\[dv_t = \theta (S_t, v_t, t) dt + \eta \omega (S_t, v_t, t) \sqrt{v_t} dZ_2,\]  
(3.2)

\[\langle dZ_1, dZ_2 \rangle = \rho dt.\]  
(3.3)

It is interesting to note how the sign of $\rho$ affects the shape of the graph of the probability density of the value of the risky asset. If $\rho$ is positive then we are saying that there is a positive correlation between volatility and value. Such a correlation would cause a heavy right tail in the graph. This is intuitively unlikely as investors will not favour an asset just because it is volatile, in fact if investors are risk averse we expect $\rho$ to be negative.
We follow the method of Gatheral, (Gatheral, 2006, p4) who in turn follows the method of Wilmott, (Wilmott, 2006, p854). Consider a portfolio of value $\Pi$ made of one unit of the option being priced which has value $V(S, v, t)$, $-\Delta$ units of the risky asset each of which is worth $S$, and a quantity $-\Delta_1$ of another asset whose value $V_1$ depends on the volatility of the risky asset. Then,

$$
\Pi = V - \Delta S - \Delta_1 V_1.
$$

The change in the value of this portfolio from time $t$ to time $t + dt$ is $d\Pi$, using Itô’s Lemma from (1.2) leads to,

$$
d\Pi = \frac{\delta V}{\delta t} dt + \frac{\delta V}{\delta S} dS + \frac{1}{2} \frac{\delta^2 V}{\delta S^2} (dS)^2 + \frac{\delta V}{\delta v} dv + \frac{1}{2} \frac{\delta^2 V}{\delta v^2} (dv)^2 + \frac{1}{2} \frac{\delta^2 V}{\delta v \delta S} dv dS.
$$

Using equations (3.1), (3.2) and (3.3) we have,

$$
d\Pi = \left\{ \frac{\delta V}{\delta t} + vS^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2} \eta^2 v^2 \frac{\delta^2 V}{\delta v^2} + \frac{1}{2} \eta \omega v S \frac{\delta^2 V}{\delta v \delta S} \right\} dt \\
- \Delta_1 \left\{ \frac{\delta V_1}{\delta t} + vS^2 \frac{\delta^2 V_1}{\delta S^2} + \frac{1}{2} \eta^2 v^2 \frac{\delta^2 V_1}{\delta v^2} + \frac{1}{2} \eta \omega v S \frac{\delta^2 V_1}{\delta v \delta S} \right\} dt \\
+ \left\{ \frac{\delta V}{\delta S} - \Delta_1 \frac{\delta V_1}{\delta S} - \Delta \right\} dS \\
+ \left\{ \frac{\delta V}{\delta v} - \Delta_1 \frac{\delta V_1}{\delta v} \right\} dv.
$$

If this is supposed to be instantaneously risk free, the $dS$ and $dv$ terms must vanish as these are random, that is

$$
\frac{\delta V}{\delta S} - \Delta_1 \frac{\delta V_1}{\delta v} = 0
$$

(3.4)
and

\[ \frac{\delta V}{\delta v} - \frac{\delta V_1}{\delta v} \Delta_1 = 0. \quad (3.5) \]

Therefore we have

\[
d\Pi = \left\{ \frac{\delta V}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} + \rho \eta \omega S \frac{\delta^2 V}{\delta v \delta S} + \frac{1}{2} \eta^2 \nu \omega^2 \frac{\delta^2 V}{\delta \nu^2} \right\} dt + \Delta_1 \left\{ \frac{\delta V_1}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V_1}{\delta S^2} + \rho \eta \omega S \frac{\delta^2 V_1}{\delta v \delta S} + \frac{1}{2} \eta^2 \nu \omega^2 \frac{\delta^2 V_1}{\delta \nu^2} \right\} dt. \quad (3.6) \]

Since this is a risk free portfolio we can only expect the risk free interest rate, hence,

\[
d\Pi = r\Pi dt = r(V - \Delta S - \Delta_1 V_1) dt. \]

Using (3.4) and (3.5) we have

\[
d\Pi = \left\{ rV - rS \frac{\delta V}{\delta S} + rS \frac{\delta V}{\delta \nu} \left( \frac{\delta V}{\delta v} \right) - rV_1 \frac{\delta V}{\delta \nu} \left( \frac{\delta V}{\delta v} \right) \right\} dt \]

Using this equation and (3.6) we arrive at,

\[
= rS \left( \frac{\delta V}{\delta v} \right) - rV_1 \left( \frac{\delta V_1}{\delta v} \right) + \left( \frac{\delta V}{\delta v} \right) \left( \frac{\delta V_1}{\delta v} \right) \left\{ \frac{\delta V}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} + \rho \eta \omega S \frac{\delta^2 V}{\delta v \delta S} + \frac{1}{2} \eta^2 \nu \omega^2 \frac{\delta^2 V}{\delta \nu^2} \right\} - rV + rS \frac{\delta V}{\delta S} \]

We now use a common technique where we gather all the \( V \) terms to one side of an equation and the \( V_1 \) terms to the other side. Since \( V \) and \( V_1 \) are independent the two sides of the equation must be equal to the same function of \( S_t, v_t \) and \( t \)
so,

\[
\frac{1}{\left(\frac{\delta V}{\delta v}\right)} \left\{ \frac{\delta V}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} + \rho \eta v \omega S \frac{\delta^2 V}{\delta v \delta S} + \frac{1}{2} \eta^2 v \omega^2 \frac{\delta^2 V}{\delta v^2} + r S \frac{\delta V}{\delta S} - r V \right\}
\]

\[
= \frac{1}{\left(\frac{\delta V}{\delta v}\right)} \left\{ \frac{\delta V_1}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V_1}{\delta S^2} + \rho \eta v \omega S \frac{\delta^2 V_1}{\delta v \delta S} + \frac{1}{2} \eta^2 v \omega^2 \frac{\delta^2 V}{\delta v^2} + r S \frac{\delta V_1}{\delta S} - r V_1 \right\}.
\]

We may use \(\theta\) and \(\omega\) from (3.2) to obtain

\[
\frac{\delta V}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} + \rho \eta v \omega S \frac{\delta^2 V}{\delta v \delta S} + \frac{1}{2} \eta^2 v \omega^2 \frac{\delta^2 V}{\delta v^2} + r \frac{\delta V}{\delta S} - r V
\]

\[
= -(\theta - \phi \omega \sqrt{v}) \frac{\delta V}{\delta v}.
\]

(3.7)

We have deliberately chosen \(\phi\) to be what Gatheral calls the market price of volatility risk (Gatheral, 2006, p6). For the explanation of the term ‘market price of volatility risk’ consider a portfolio of value \(\Pi_1\), consisting of an option of value \(V\) and \(-\frac{\delta V}{\delta S}\) units of risky asset, so that,

\[
\Pi_1 = V - \frac{\delta V}{\delta S}.
\]

Then applying Itô’s Lemma as above, we have

\[
d\Pi_1 = \left\{ \frac{\delta V}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} + \rho \eta v \omega S \frac{\delta^2 V}{\delta v \delta S} + \frac{1}{2} \eta^2 v \omega^2 \frac{\delta^2 V}{\delta v^2} \right\} dt + \left\{ \frac{\delta V}{\delta S} - \Delta \right\} dS + \left\{ \frac{\delta V}{\delta v} \right\} dv.
\]
As the option is delta hedged, we assume the $dS$ coefficient is zero. We now calculate the difference between the change in the value of the portfolio $d\Pi_1$, and the change we expect if it was a risk free investment, namely,

$$d\Pi_1 - r(\Pi)dt = d\Pi_1 - rVdt + r \frac{\delta V}{\delta S} Sdt$$

$$= \left\{ \frac{\delta V}{\delta t} + \frac{1}{2} \nu S^2 \frac{\delta^2 V}{\delta S^2} + \rho \nu \omega S \frac{\delta^2 V}{\delta v \delta S} + \frac{1}{2} \eta^2 \nu \omega^2 \frac{\delta^2 V}{\delta v^2} - rV + rS \frac{\delta V}{\delta S} \right\} dt + \left\{ \frac{\delta V}{\delta v} \right\} dv$$

We use (3.7) to show

$$= -(\theta - \phi \omega \sqrt{v}) \frac{\delta V}{\delta v} dt + \frac{\delta V}{\delta v} dv$$

and we use (3.2) to obtain

$$= \phi \omega \sqrt{v} \frac{\delta V}{\delta v} dt - \frac{\delta V}{\delta v} (dv - \eta \omega \sqrt{v} dZ_2) + \frac{\delta V}{\delta v} dv$$

$$d\Pi_1 - r(\Pi)dt = \omega \sqrt{v} \frac{\delta V}{\delta v} (\phi dt + \eta dZ_2)$$

This gives us $\phi$ as the excess return above the risk free interest rate from $\Pi_1$. That is, we would expect to receive the $\eta dZ_2$ part, but the $\phi$ is above what we would get from a constant volatility model. So we have an expression for the return due to the market price of volatility. (Note typographical error in the last equation on p7 of Gatheral, 2006)
3.1 Derivation of the Dupire Eqn for Local Volatility

A local volatility model treats the volatility, $\sigma$, as a function of the asset price, and the time. It is not, itself, stochastic; these models are simpler to use than the stochastic model. We begin with (2.1), and we follow the work of Derman and Kani, (Derman and Kani, 1994, p39).

We begin with the following stochastic differential equation to describe the price of the risky asset $S_t$, using the usual variables from (2.1),

$$
\frac{dS_t}{S_t} = rdt + \sigma dW_t.
$$

(2.1)

Now consider the following where we let,

$\psi$ be the probability density of the stock price $S$ at time $T$,

$K$ be the strike price and

$D$ be the discount factor, meaning that $D = e^{-\int_0^T r(t)dt}$. Then the risk neutral value of a plain vanilla European option, $C(S_0, K, T)$ is given at time $t = 0$ by,

$$
C(S, K, T) = D \int_{\infty}^{\infty} \psi(S', T, S)(S' - K)dS'.
$$

(3.8)

Differentiating (3.8) with respect to $K$ we have,

$$
\frac{\delta C}{\delta K} = -D \int_{K}^{\infty} \psi(S', T, S)dS'.
$$
Differentiating again with respect to $K$ we get,

$$\frac{\delta^2 C}{\delta K^2} = D \psi(S', T'S').$$

Next we calculate the time rate of change of $C$ from (3.8).

$$\frac{\delta C}{\delta T} = -r(T)C + D \int \frac{\delta}{\delta T} \left[ \psi(S', T, S) \right] (S' - K) dS'.$$

Applying the Fokker Planck equation (as used also by Gatheral (Gatheral, 2006, p10)),

$$\frac{\delta \psi}{\delta T} = \frac{1}{2} \frac{\delta^2}{\delta S'^2} \left( \sigma^2 S'^2 \psi \right) - \frac{\delta}{\delta S'} (rS' \psi)$$

we obtain,

$$\frac{\delta C}{\delta T} = -r(t)C + D \int_K^\infty \left\{ \frac{1}{2} \frac{\delta^2}{\delta S'^2} \left( \sigma^2 S'^2 \psi \right) - \frac{\delta}{\delta S'} (rS' \psi) \right\} (S' - K) dS'.$$

Integrating by parts,

$$\frac{\delta C}{\delta T} = -r(T)C + D \left[ (S' - K) \frac{\delta}{\delta S'} (\sigma^2 S'^2 \psi) \right]_K^\infty - D \int_K^\infty \frac{\delta}{\delta S'} (\sigma^2 S'^2 \psi) dS'$$

$$- D [(S' - K)rS' \psi]_K^\infty + D \int_K^\infty rS' \psi dS'.$$
we can assume $\psi \to 0$ as $S' \to \infty$ fast enough so that $S' \psi \to 0$ as $S' \to 0$, thus

$$= -r(T) + \frac{\sigma^2 K^2}{2} D\psi + rD \int_{K}^{\infty} S' \psi dS'$$

$$= -r(T)D \int_{K}^{\infty} \psi(S' - K)dS' + \frac{\sigma^2 K^2}{2} D\psi + rD \int_{K}^{\infty} \psi dS'$$

$$= \frac{\sigma^2 K^2}{2} D\psi + rD \int_{K}^{\infty} K \psi dS'$$

$$\frac{\delta C}{\delta T} = \frac{\sigma^2 K^2 \delta^2 C}{2} \frac{\delta^2 C}{\delta K^2} - r(T)K \frac{\delta C}{\delta K}.$$  \hfill (3.10)

So far we have used $C(S_0, K, T)$, now it is more convenient to consider $C_F(F_T, K, T)$. This is the value of a call using the forward price of the asset $F_T$, where

$$F_T = S_0 e^{\int_0^T \mu(t)}.$$

Under a risk neutral measure which we will use to price options, we may rearrange (3.10) to obtain a definition of the local volatility, $\sigma$, where we may estimate $\frac{\delta C}{\delta T}$ and $\frac{\delta^2 C}{\delta K^2}$ from the prices of the risky asset. This is effectively a definition of local volatility, $\sigma$.

$$\sigma^2 = \frac{\frac{\delta C_F}{\delta T}}{2 \frac{\delta^2 C_F}{\delta K^2}}$$  \hfill (3.11)
3.2 Local Variance in Terms of Instantaneous Variance

We follow Gatheral (Gatheral, 2006, p13) who follows (Derman and Kani, 1998). Extending the definition of the forward price to a general time \( t, 0 \leq t \leq T \), we have

\[
F_{t,T} = S_t e\int_t^T \mu ds
\]

thus

\[
dF_{t,T} = \sqrt{\nu_t} F_{t,T} dZ = dS_T.
\]

Now, we consider \( C(S_0, K, T) \), the undiscounted value of a call with strike \( K \) and expiry \( T \),

\[
C(S_0, K, T) = E \left[ (S_T - K)^+ \right],
\]

where \( E \) denotes the risk neutral expectation. Differentiating with respect to \( K \) gives,

\[
\frac{\delta C}{\delta K} = -E \left[ H(S_T - K) \right]
\]

where \( H \) is the Heaviside step function. Differentiating again we get,

\[
\frac{\delta^2 C}{\delta K^2} = E \left[ \delta(S_T - K) \right],
\]

where \( \delta \) is the Dirac delta function. We now apply Itô’s Lemma (1.1) to
\((S_T - K)\) and use the above calculations,

\[
d(S_T - K)^+ = \frac{\delta}{\delta K} \{ (S_T - K)^+ \} dK + \frac{\delta}{\delta S_T} \{ (S_T - K)^+ \} dS_T + \frac{1}{2} \frac{\delta^2}{\delta S_T^2} \{ (S_T - K)^+ \} (dS_T)^2
\]

\[
= 0 + H(S_T - K)dS_T + \frac{1}{2} v_T S_T^2 \delta(S_T - K)dT,
\]

and taking expectations conditional on \(S_t = K\) we have,

\[
dE [(S_T - K)^+] = dC = \frac{1}{2} E [v_T S_T^2 \delta(S_T - K)] dT,
\]

\[
= \frac{1}{2} K^2 \frac{\delta^2 C}{\delta K^2} E [v_T | S_T = K].
\]

Hence

\[
\frac{\delta C}{\delta T} = \frac{1}{2} K^2 \frac{\delta^2 C}{\delta K^2} E [v_T | S_T = K]
\]

Comparing this with (3.11) we see

\[
\sigma^2 = E [v_T | S_T = K]. \tag{3.12}
\]

We now have two important equations (3.11) and (3.12), the first gives a definition of the local volatility \(\sigma\) which is measurable from the data, and the second connects this to the expectation of the stochastic variance \(\bar{v}\).

A problem with the SV model is that the real market is incomplete (Cont and
Tankov, 2004, p14) and that the hedging strategy we use is a somewhat theoretical, as it depends on the existence of an asset whose value depends on the volatility of the underlying. We should therefore proceed with caution. It is more realistic to say that a model can explain what has happened than to think it can tell us what will happen.
4 Heston Model

The assumption of the Heston Model is that the stochastic volatility is mean reverting. We make this assumption because it is unreasonable to think that the volatility will settle to become zero, and unthinkable to reason that it would be infinite. That is, we do not expect risky assets to stop being risky which would exclude zero volatility, and we do not wish to contemplate the possibility that the risk would be infinite. Thus a simple non constant model is to assume that the volatility varies about a mean, furthermore we propose that it follows a square root diffusion.

Defining

\( v_t \) to be the instantaneous variance of the value of the risky asset,
\( \bar{v} \) to be the mean of \( v_t \),
\( \lambda \) to be the speed of reversion of the \( v_t \) to \( \bar{v} \),
\( \eta \) to be the variance of \( v_t \) and with

\( Z_1 \) and \( Z_2 \) standard Brownian motions with correlation \( \rho \), the price dynamics are assumed to be described by the following equations (4.1),(4.2) and (4.3).

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu_t dt + \sqrt{v_t} S_t dZ_1 \\
\frac{dv_t}{v_t} &= -\lambda (v_t - \bar{v}) dt + \eta \sqrt{v_t} dZ_2 \\
\langle dZ_1, dZ_2 \rangle &= \rho dt
\end{align*}
\]

Substituting for \( \theta \) and \( \omega \) into (3.7) we obtain,
\[
\frac{\delta V}{\delta t} + \frac{1}{2} v S^2 \delta^2 V \delta S^2 + \rho \eta v S \delta^2 V \delta v \delta S + \frac{1}{2} \eta^2 v \delta^2 V \delta v^2 + rS \delta V \delta S - rV = \lambda(v - \bar{v}) \frac{\delta V}{\delta v} \quad (4.4)
\]

We now wish to solve this equation to obtain an explicit solution.

### 4.1 An Explicit Solution

We follow Gatheral (Gatheral, 2006, p16) who follows Heston (Heston, 1993, p328).

We follow the usual terminology and let,

- \( K \) be the strike price of the option and
- \( T \) be the time to expiry. Then we define

\( F_{t,T} \) to be the time \( T \) forward price of the stock index, where

\[
F_{t,T} = S_t e^{\int_t^T \mu \, ds}.
\]

Then we use a new variable \( x \) where \( x = \log \frac{F_{t,T}}{K} \) and we let

\( C_F \) be the future value at expiration of the European option (note this is not usually its value today) and as is usual we let \( \tau = T - t \) the time remaining until expiry.

Then (4.4) gives

\[
- \frac{\delta C_F}{\delta \tau} + \frac{\delta^2 C_F}{\delta x^2} \frac{v}{2} - \frac{\delta C_F}{\delta x} \frac{v}{2} + \frac{\delta^2 C_F}{\delta v^2} \frac{\eta^2 v}{2} + \frac{\delta^2 C_F}{\delta x \delta v} \rho \eta v = \lambda(v - \bar{v}) \frac{\delta C_F}{\delta v} \quad (4.5)
\]

Duffie, Pan and Singleton (Duffie, Pan and Singleton, 2000) state that a solution of (4.5) has the form

\[
C_F(x, v, t) = K \left\{ (e^x P_1(x, v, t) - P_0(x, v, t)) \right\}. \quad (4.6)
\]
We calculate each of the terms of (4.5) using (4.6). In particular we are careful with the terms,

\[ \frac{\delta C_F}{\delta x} = K \left[ e^x P_1 + e^x \frac{\delta P_1}{\delta x} - \frac{\delta P_0}{\delta x} \right] \]

and the term

\[ \frac{\delta^2 C_F}{\delta x^2} = K \left[ e^x P_1 + e^x \frac{\delta P_1}{\delta x} + e^x \frac{\delta P_1}{\delta x} + e^x \frac{\delta^2 P_1}{\delta x^2} - \frac{\delta^2 P_0}{\delta x^2} \right], \]

because these terms introduce \( j \) as a variable in eqn (4.7) and not just as a label.

The system may be condensed into an equation in \( P_j \), for \( j = 0, 1 \), since \( 1 \) and \( e^x \) are linearly independent. We set new variables \( a \) and \( b_j \), where

\[ a = \lambda \tilde{v} \text{ and } b_j = \lambda - j \eta \rho, \]

and have,

\[ -\frac{\delta P_j}{\delta \tau} + \frac{1}{2} \frac{\delta^2 P_j}{\delta x^2} v - \frac{\delta P_j}{\delta x} \left( \frac{1}{2} - j \right) v + \frac{1}{2} \eta^2 v \frac{\delta^2 P_j}{\delta v^2} + \rho \eta v \frac{\delta^2 P_j}{\delta x \delta v} + (a - b_j v) \frac{\delta P_j}{\delta v} = 0. \]  

(4.7)

We have the following boundary condition which is due to the value of the call option at expiry. Namely,

\[ \lim_{\tau \to 0} P_j(x, v, t) = H(x), \]

where \( H \) is the Heaviside step function, because if the underlying is above the strike price at expiry when \( \tau = 0 \), the option is in the money, if not the option is
worthless.

To solve (4.7) we use a Fourier transform technique. We denote the Fourier transform of \( P_j \) by \( \tilde{P}_j \), and use \( \sqrt{-1} = i \), then

\[
\tilde{P}_j(u, v, \tau) = \int_{-\infty}^{\infty} e^{-iux} P_j(x, v, t) \, dx,
\]

and the inverse transform is given by,

\[
P_j(x, v, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} \tilde{P}_j(u, v, t) \, du.
\]

Let us recall the properties of the Fourier transform (Dettman, 1984, p371). Denote by \( \tilde{f} \) the Fourier transform of the function \( f \), which has continuous derivatives \( f^{(k)} \) for \( k = 0, ..., n - 1 \), and piecewise continuous derivative \( f^{(n)} \). Then we have \( \tilde{f}' = iz \tilde{f} \), and \( f^{(n)} = (iz)^n \tilde{f} \).

Hence we have,

\[
\frac{\delta \tilde{P}_j}{\delta \tau} = \frac{\delta}{\delta \tau} (\tilde{P}_j), \quad \frac{\delta \tilde{P}_j}{\delta x} = iu \frac{\delta}{\delta x} (\tilde{P}_j), \quad \frac{\delta^2 \tilde{P}_j}{\delta x^2} = \nu^2 u^2 \frac{\delta^2}{\delta x^2} (\tilde{P}_j),
\]

\[
\frac{\delta \tilde{P}_j}{\delta v} = \frac{\delta}{\delta v} (\tilde{P}_j), \quad \frac{\delta \tilde{P}_j}{\delta v^2} = \frac{\delta}{\delta v^2} (\tilde{P}_j), \quad \frac{\delta^2 \tilde{P}_j}{\delta v \delta x} = iu \frac{\delta^2}{\delta v \delta x} (\tilde{P}_j),
\]

so that the Fourier transform of (4.7) is,

\[
-\frac{\delta \tilde{P}_j}{\delta \tau} + \frac{1}{2} \nu (-u^2) \tilde{P}_j - \left( \frac{1}{2} - j \right) \nu u \tilde{P}_j + \frac{1}{2} \eta^2 v \frac{\delta \tilde{P}_j}{\delta v^2} + \rho \eta v i u \frac{\delta \tilde{P}_j}{\delta v} + (a - b j) \frac{\delta \tilde{P}_j}{\delta v} = 0.
\]
Defining $\alpha$, $\beta$ and $\gamma$,

$$
\alpha = -\frac{u^2}{2} - \frac{iu}{2} + iju, \quad \beta = \lambda - \rho \eta j - \rho \eta ju, \quad \gamma = \frac{\eta^2}{2}.
$$

(4.8)

we have

$$
v \left\{ \alpha \delta \tilde{P}_j - \beta \frac{\delta \tilde{P}_j}{\delta v} + \gamma \frac{\delta^2 \tilde{P}_j}{\delta v^2} \right\} + a \frac{\delta \tilde{P}_j}{\delta v} - \frac{\delta \tilde{P}_j}{\delta \tau} = 0.
$$

(4.9)

We use the following form of solution as proposed by Gatheral (Gatheral, 2006, p18)

$$
\tilde{P}_j(u, v, \tau) = \tilde{P}_j(u, v, 0)e^{[F(u,t)\bar{v}+D(u,t)v]}
= \frac{1}{iu}e^{[F(u,t)\bar{v}+D(u,t)v]},
$$

where

$$
\tilde{P}_j(u, v, 0) = \int_{-\infty}^{\infty} e^{-iux}H(x)dx = \frac{1}{iu}.
$$

We now calculate $\frac{\delta \tilde{P}_j}{\delta \tau}$, $\frac{\delta \tilde{P}_j}{\delta v}$ and $\frac{\delta^2 \tilde{P}_j}{\delta v^2}$.

First,

$$
\frac{\delta \tilde{P}_j}{\delta \tau} = \frac{1}{iu}e^{F\bar{v}+Dv} \left( \frac{\delta F}{\delta \tau} \bar{v} + \frac{\delta D}{\delta \tau} v \right) = \left\{ \bar{v} \frac{\delta F}{\delta \tau} + v \frac{\delta D}{\delta \tau} \right\} \tilde{P}_j,
$$

(4.10)

next,

$$
\frac{\delta \tilde{P}_j}{\delta v} = \frac{1}{iu}e^{F\bar{v}+Dv} D \tilde{P}_j,
$$

(4.11)
and finally,

\[ \frac{\delta^2 \tilde{P}_j}{\delta v^2} = D^2 \frac{1}{i u} e^{F_{\bar{v}} + D v} + \left( \frac{1}{i u} e^{F_{\bar{v}} + D v} \frac{\delta D}{\delta v} \right). \]

but

\[ \frac{\delta D}{\delta v} = 0 \]

so,

\[ \frac{\delta^2 \tilde{P}_j}{\delta v^2} = D^2 \tilde{P}_j. \]  \hspace{1cm} (4.12)

We substitute (4.10), (4.11) and (4.12) into (4.9) to obtain,

\[ v \left\{ \alpha \tilde{P}_j - \beta D \tilde{P}_j + \gamma D^2 \tilde{P}_j \right\} + a D \tilde{P}_j - \tilde{P}_j \left( \bar{v} \frac{\delta F}{\delta \tau} + v \frac{\delta D}{\delta \tau} \right) = 0 \]

\[ \iff \quad v \left\{ \alpha - \beta D + \gamma D^2 \right\} = \bar{v} \frac{\delta F}{\delta \tau} + v \frac{\delta D}{\delta \tau} - a D. \]  \hspace{1cm} (4.13)

Then we see that since \( a = \bar{v} \lambda \), a solution to (4.13) is given by,

\[ \frac{\delta F}{\delta \tau} = \lambda D \]  \hspace{1cm} (4.14)

and

\[ \frac{\delta D}{\delta \tau} = \alpha - \beta D + \gamma D^2. \]
We factorize the second of these equations by the usual formula for quadratics giving the roots \( r_\pm \),

\[
r_\pm = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha \gamma}}{2\gamma}.\]

We integrate (4.15) using partial fractions to obtain \( D \) and hence \( F \), the algebraic manipulation has been verified using Maple (see appendix for the code).

We set \( g = \frac{r_+}{r_\pm} \) and \( d = \sqrt{\beta^2 - 4\alpha \gamma} \) and hence,

\[
\frac{\delta D}{\delta \tau} = \gamma(D - r_+)(D - r_-), \tag{4.15}
\]

this gives the solutions,

\[
D(u, \tau) = r_- \left\{ \frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}} \right\} \tag{4.16}
\]

and

\[
F(u, \tau) = \lambda \left\{ r_+ \tau - \frac{2}{\eta^2} \log \left( \frac{e^{d\tau} - g}{1 - g} \right) \right\} \tag{4.17}
\]

The inverse transform gives the following for \( P_j(x, v, \tau) \)

\[
P_j(x, v, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left\{ \frac{e^{[F_j(u, \tau)v + D_j(u, \tau)v + iux]}}{iu} \right\} du \tag{4.18}
\]

We now have an explicit solution when the above integral (4.18) is substituted into (4.6), using (4.16) and (4.17). However to avoid a problem with a complex logarithm we use (4.19) instead of (4.17). This is dealt with in the next subsection.
4.2 A Complex Logarithm Conjecture

The equation for $F(u, \tau)$, (4.17) uses the complex logarithm, it turns out that it is better to use (4.19). The complex logarithm function sends $x + iy = r(\cos \theta + i \sin \theta) \rightarrow r + i\theta$. The choice of the range of $\theta$, is of concern. The principle branch of logarithm uses $\theta \in (-\pi, \pi]$ but another choice could be $\theta \in [0, 2\pi)$. These are different functions. We need to be careful which branch of logarithm we use. Numerical methods normally use the principle branch which cuts the complex plane along negative Real axis. In fact there are many choices of where to cut the complex plane, and the cut need not even be a straight line. The problem is that the logarithm is not continuous across this cut, and therefore numerical methods will be not be reliable if the path of the argument of the logarithm crosses the negative Real axis. Gatheral says that using (4.19) avoids the problem (Gatheral, 2006, p19-20).

$$F(u, \tau) = \lambda \left\{ r\tau - \frac{2}{\eta^2} \log \left( \frac{1 - ge^{-d\tau}}{1 - g} \right) \right\}. \quad (4.19)$$
4.3 Numerical Integration

We present the results of the explicit solution to the Heston model’s in table 4.1. We use the parameters for the 15th September 2005 from Gatheral (Gatheral, 2006, p40) to calculate the prices of the call options. The prices for puts were calculated using the put call parity relationship (1.3). The explicit solution does not give the present value of the call, but gives the future value. Similarly it uses not the present value of the asset but the time $T$ forward value. This means that when we use the put call parity relationship we will not discount the strike price, and the result will be the forward value of the put. In this case there is little point applying the discount to the resulting put prices as the time period is tiny.

We find that for $\frac{10}{12}$ in the money calls and $\frac{3}{5}$ in the money puts the explicit solutions are inside the bid ask spread. The picture is different for out of the money options, none of the explicit solutions for out of the money options are inside the bid ask spread. Of the explicit solutions for calls, $\frac{2}{5}$ are below the bid ask spread and $\frac{6}{12}$ of the puts are below. These results are generally better than the BSM formula solutions in table 2.1.

4.3.1 Evidence to Support the Accuracy of Numerical Integration

It was necessary to use a numerical method to evaluate the integral in (4.18) and produce the results in table 4.1 on page 16. Since the limits of (4.18) are zero and infinity we have to check that the integrand goes to zero fast enough so that we may pick a reasonable upper limit for the numerical integration. This was assumed in the derivation of (3.9). We also need to check that the integrand is well behaved near zero, that is, we wish to verify that our method for numerical integration
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Table 4.1: Heston Explicit solutions for SPX options on 15th September 2005 and actual prices. There was no information for strikes of 1165, 1185 and 1255.

will converge. While numerical methods do not prove either convergence or good behaviour at zero, the plots in figures 4.1, 4.2 and 4.3 lend considerable credibility to the assumption.

Let us consider that for each evaluation of the numerical integral of (4.18), $\tau$ and $v$ are constant. Then define,

$$Q(j, x, u) = Re \left\{ e^{i(u, j) \bar{\xi} + D(u, j)\nu + iux} \right\}.$$

Using this we see that for $C(x)$, the value of a call option and $x = \log \frac{F_t}{K}$,
(4.6) becomes,

\[ C(x) = \frac{1}{\pi} K \left( \int_0^\infty e^x Q(1, x, u) - Q(0, x, u) du \right) + \frac{1}{2} (F_{t,T} - K) \]

It is clear that \((F_{t,T} - K)\) does not vary with the integration variable \(u\), therefore the convergence of (4.18) and hence the reliability of the numerical estimation of (4.6) depends on the behaviour of \(M(K, u)\), where

\[ M(K, u) = K \left( e^x Q(1, x, u) - Q(0, x, u) \right). \]

It is seen in figure 4.1 that this function is almost zero for \(u > 300\) and we see in figure 4.2 that it does not display erratic behaviour as \(u \to 0\). It is seen from figure 4.3 that for a strike price exactly at the money, \(M\) does not oscillate. The oscillations are important, the gap used in the numerical integration must be considerably smaller than the wavelength of these oscillations, hence we use 0.001. Since \(M(K,1000) < 10^{-14}\) for for several values of \(K\), the upper limit of 1000 is used for the numerical integration.
Fig. 4.1: This shows that the $M(1160, u)$ function goes to zero quickly. It is very interesting between 50 and 250, and is effectively zero beyond 300.

4.4 Monte Carlo Simulation using Milstein Discretization

The explicit solution relies on the numerical estimation of an integral. We now examine another way to obtain a numerical solution to (4.1),(4.2) and (4.3) using the Monte Carlo Method, this has the advantage that it can be easily adapted to estimate the prices of exotic options. We leave the estimation of the parameters $v_t, \bar{v}, \lambda, \eta$ and $\rho$ to the next subsection.

Here we outline the Milstein discretization as quoted by Gatheral (Gatheral, 2006, p22). The Euler discretization can allow negative variances and is very
Fig. 4.2: This shows the behaviour of the $M(1160, u)$ function nearer to zero, and that it would seem to tend to a finite limit at zero.

slow to converge. The Milstein discretization is essentially found by using the Itô Lemma to a higher order. For the discretized variance we have,

$$v_{i+1} = v_i - \lambda(v_i - \bar{v}) + \eta \sqrt{v_i \Delta t} Z + \frac{\eta^2}{4} \Delta t \left(Z^2 - 1\right).$$

This can be rewritten as,

$$v_{i+1} = \left(\sqrt{v_i} + \frac{\eta}{2} \sqrt{\Delta t} Z\right)^2 - \lambda(v_i - \bar{v}) \Delta t - \frac{\eta^2}{4} \Delta t.$$
Fig. 4.3: This shows the behaviour of the $M(1227.73, u)$ function. Note that it does not display oscillations as $M(1160, u)$ did, but does tend to zero whilst positive.

For the price of the risky asset we have,

$$x_{i+1} = x_i - \frac{v_i}{2} \Delta t + \sqrt{v_i \Delta t} W,$$

where $x_i = \log \frac{S_i}{S_0}$. $W$ and $Z$ are standard normally distributed random variables with correlation $\rho$.

The difficulty is that we may still have a negative variance due to discretization error. In the Matlab code, used in Milstein.m, we avoid this by using absolute value of the variance (see appendix). In order to demonstrate the usefulness of
the Milstein discretization we present the results using the parameters of from Gatheral’s Table 3.2 (Gatheral 2006, p40) and the actual prices of the calls and puts for that day (Gatheral 2006, p51) in Table 4.2.

This date, 15th September 2005, is chosen as it is one day before the expiry of the options. It will demonstrate that the prices are higher than one would expect using the straightforward Heston model. The actual prices for these options are displayed (Gatheral, 2006, p51). There is of course a word of warning in this method, namely that short expiry options are very likely to be affected by liquidity concerns. However the point of the exercise is to show the eventual need for further development beyond Heston’s model.

The Monte Carlo (MC) programme, Milstein.m, produced Table 4.2 using \( \Delta t = 10^{-5} \), \( T = \frac{1}{250} \) and \( N = 5,000 \) iterations, this is equivalent to recalculating the process every five minutes for a trading day. The 95 naturally with a Monte Carlo process we wish to check convergence, we did this by repeating the calculations with \( N = 20,000 \) and \( \Delta t = 10^{-6} \); the running time went from 20 minutes to 2 hours. The results were practically identical with the same pattern of overlap between the bid ask spread and the confidence intervals.

In order to check the working of Milstein.m, we use the parameters of the Heston Nandi model (Gatheral, 2006, p44) and compare the output with Fig 4.1 of Gatheral (Gatheral, 2006, p45). The results are presented in a histogram figure 4.4. The parameters used for Heston Nandi model are \( v = 0.04; \bar{v} = 0.04; \lambda = 10, \eta = 1 \) and \( \rho = -1 \). It is notable that the shape of Fig 4.4 is basically the same as the shape of Gatheral’s Fig 4.1 (Gatheral, 2006, p45), though shifted a little to the left. We conclude that the Milstein discretization is working quite well.
<table>
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<th>Call Milstein</th>
<th>Call Ask</th>
<th>C.I.</th>
<th>Strike</th>
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</table>

Table 4.2: Comparison of Milstein calculations and actual prices for short expiry SPX options. The programme used $\Delta t = 10^{-5}$ and $T = \frac{1}{250}$ and $N = 5,000$ iterations.

### 4.5 The Predicted Implied Volatility Surface

We look at some properties of the volatility surface predicted by the Heston model. It is to be hoped that this surface will coincide with the observed values of $\sigma_{imp}(K,T)$. There are of course practical problems. First there is only ever a discrete list of prices available from which too calculate $\sigma_{imp}$; and second, the prices are quoted for bid and ask, so we really have two values of $\sigma_{imp}$ for each pair $(K,T)$. The hope would be that any theoretical surface would fit between these pairs of values.

Considering the general SV model Gatheral (Gatheral, 2006, p30) shows that
Fig. 4.4: A histogram of the Milstein Discretization of Heston Nandi Model showing log return.

if we let,

\[ \sigma^2_{imp}, \] be the Black Scholes implied variance,

\[ K, \] be the strike price

\[ T, \] be the expiry and

\[ v_L, \] be the local variance; and if we introduce

\[ \bar{x}_t, \] the most likely path of \( x_t \) starting at the initial price at time \( t = 0 \) and ending at the strike price at time \( T \) (Gatheral, 2006, p30), assuming that \( S_T = K \)
then we have,

\[ \sigma_{imp}^2 \approx \frac{1}{T} \int_0^T v_L(\tilde{x}_t) dt. \]  \hspace{1cm} (4.20)

This is summarized very well by Gatheral:

we now have a very simple and intuitive picture for the meaning of Black-Scholes implied variance of a European option with a given strike and expiration: It is approximately the integral from today to expiration of local variances along the most probable path for the stock price conditional on the stock price at expiration being the strike price of the option. (Gatheral, 2006, p31)

Applying this to the Heston model where

\( v, \) is the instantaneous variance

\( \lambda' = \lambda - \frac{\rho \eta}{2} \) where \( \bar{v} \) is the mean variance of the asset,

\( \bar{v}' = \frac{\bar{v}\lambda}{\lambda'} \)

this yields (4.21),

\[ \sigma_{imp}^2 \bigg|_{K=F_T} \approx (v - \bar{v}') \frac{1 - e^{-\lambda'T}}{\lambda'T} + \bar{v}'. \]  \hspace{1cm} (4.21)

This has the consequence that

\[ \sigma_{imp}^2 \bigg|_{K=F_T} \rightarrow v, \quad T \rightarrow 0 \]
\[ \sigma_{imp}^2 \mid K = F_T \to \bar{v}, \quad T \to \infty \]

as one would expect.

It is more interesting to see how the implied volatility changes with \( \rho \) and \( \eta \). It turns out that the dependence is roughly linear. We have from (Gatheral, 2006, p35)

\[ \frac{\delta}{\delta x_t} \sigma_{imp}^2 = \frac{\rho \eta}{\lambda T} \left\{ 1 - \frac{1 - e^{-\lambda T}}{\lambda T} \right\} \] (4.22)

therefore,

\[ \frac{\delta}{\delta x_t} \sigma_{imp}^2 \to \frac{\rho \eta}{2}, \quad T \to 0, \]

and

\[ \frac{\delta}{\delta x_t} \sigma_{imp}^2 \to \frac{\rho \eta}{\lambda T}, \quad T \to \infty. \]

There are straight line asymptotes which may be used for calibration. Again, a word of warning is appropriate, options that are either very far in the money or very far out of the money are not frequently traded, this is often the case with very short or very long expiry options. Thus they are not a very reliable source of information to calculate parameters, it is better to use the prices of options which are traded in high volumes.
4.6 SVI Model

Based on the straight line asymptotes for $\sigma_{imp}$, Gatheral has suggested a straightforward model for the shape of the volatility surface. He describes this model as Stochastic Volatility Inspired (SVI). It is based on the shape of the surface and while it is not directly derived from the dynamics, it does work reasonably well.

We follow (Gatheral, 2006, p37) and (Gatheral, 2004). Gatheral proposes that for

\[ k = \log(K/S_t), \]

\[ a, \] the overall level of variance,

\[ b, \] the angle between the left and right asymptotes,

\[ s, \] a measure of the smoothness of the meeting point of the asymptotes,

\[ \rho, \] the orientation of the graph and

\[ m, \] a translation parameter, that

\[ \sigma^2_{imp} = a + b \left( \rho(k - m) + \sqrt{(k - m)^2 + \sigma^2} \right). \]

The following plot in figure 4.5 was made with \( a = 0.04, b = 0.4, s = 0.1, \rho = -0.4, m = 0 \) and it shows the asymptotes. The equations of the asymptotes are for the left and right

\[ \sigma^2_{imp} = a - b(1 - \rho)(k - m), \quad \sigma^2_{imp} = a + b(1 + \rho)(k - m). \]

4.7 Estimation of the Heston Parameters

It is very difficult to estimate the parameters with a sufficiently small error. We wish to find estimates for: \( v, \tilde{v}, \lambda, \rho \) and \( \eta \), subject to the following conditions,
Fig. 4.5: Plot of the SVI model with implied variance against $k = \log(K/S_t)$.

$v \geq 0$, as it is a variance it cannot be negative,

$\bar{v} > 0$, as a variance of a risky asset needs to be non zero,

$\eta > 0$, since the variance of the asset is stochastic,

$\lambda > 0$, as we want the volatility of the asset to revert to its mean, and

$-1 < \rho < 1$, as it is a correlation,

It is to be expected that the correlation $\rho$ would actually be negative. This is because we expect that an unexpected decrease in an asset’s value would increase the market’s perception of its volatility, (and hence increase its actual volatility).

The basic method to estimate parameters is that for a particular option we
vary the five parameters so that we equate the observed market price of the option, and the predicted value from the Heston Model. We do this simultaneously for several combinations of strike price and expiries, and minimize the sum of the errors. The process will hopefully yield a set of five parameters giving the least overall error. It is common to use the sum of the squares of the errors instead of the magnitudes of the errors. The problem with this general method is that the minimized error, however it is calculated, in the five dimensional space might only be a local minimum. To test all the possible combinations of five variables would use considerable computing power, and might not be possible for each risky asset each day. It would be considered reasonable however that once values are found that are working well, that there would only be a daily correction rather than an exhaustive search. For example Excel’s solver might be sufficient if nothing serious happens between calibrations. The advantage of the Heston model for the general problem of parameters is that we can use the explicit solution to find the parameters starting from traded option prices. We may then use these parameters in the Monte Carlo simulation to find the prices of exotic or path dependent options.

This problem is discussed in (Mikhailov and Nögel, 2003), (Bakshi,Cao and Chen, 1997) and in (Gatheral, 2006, p40).

4.8 Remarks Concerning the Heston Model

Consideration of the prices of short expiry options leads one to conclude that the observed implied volatility level cannot be fully explained by the stochastic volatility model of Heston. This leads to further developments of the model dealt
with later in this report. Rebonato suggests that the excess of implied volatility might be a consequence of the market crash on Black Monday, 17th October 1987 (Rebonato, 2004, p439). After such a shock traders may have been prepared to pay for a little ‘insurance’ even for a very short period of time as they consider the possibility that the price of a risky asset might move suddenly.
5 Further Developments, Jump Diffusion Model and SABR

We briefly look at two developments since the Heston Model. First the Jump Diffusion (SVJ) model and second the Stochastic, $\alpha, \beta, \rho$ (SABR) model.

5.1 Stochastic Volatility with Jumps (SVJ)

In the SV model risky asset prices followed a continuous process and so there was in some sense a limit to how much an asset value could be expected to move in a small time. Thus, an improvement to the SV model is to allow jumps. This will account for the higher than expected short term volatility one observes, however it raises questions about the frequency and size of these jumps. It is to be expected that the jump diffusion will not be significantly different from the stochastic volatility at large expiry times since the intervening time would allow the SV model to account for the implied volatility. However the jump diffusion model will be useful in explaining short expiry option prices.

We follow Gatheral (Gatheral, 2006, p52), who follows Wilmott (Wilmott, 2006, p931).

$S$ is the value of the risk asset,

$\mu$ is the drift,

$\sigma$ is the volatility

$Z$ is a standard Brownian motion, which is independent of the jump process,

$J$ is the proportion by which $S$ changes when a jump occurs, we assume that this is known, and
\( dq \) is a Poisson Process \( dq = 1 \) with probability \( \lambda(t)dt \), otherwise \( dq = 0 \). Then the SVJ model proposes that,

\[
dS = \mu S dt + \sigma S dZ + (J - 1)S dq
\]

For example if \( J = \frac{2}{3} \), then when \( dq = 1 \), the change in \( S \), \( dS \) is immediately \((1 - \frac{2}{3})S = -\frac{1}{3}S\), that is \( S \) changes from \( S \) to \( \frac{2}{3}S \). If \( J = 1 \), there is no jump at all. The advantage of this model is of course that it can fit the empirical data better than the SV model as it can account for the short term implied volatility. The disadvantage is that the mathematics is much more complicated. This is because in addition to the needs of the SV model we must model the distribution of the jump events and also their size. This leads to yet more parameters to be estimated and more room for error. It would seem reasonable to look for the jump parameters at short expiry where their effect is most pronounced, and to look at the longer expiry data to fit the background SV model.

5.2 Stochastic \( \alpha, \beta, \rho \) (SABR)

We base this discussion on Wilmott, (Wilmott, 2007, p292-293), with some input from (Gatheral, 2006, p91-93). This model is best suited to modeling bond yields, and other fixed income instruments which tend to have lower volatility than stocks, currencies or commodities. It proposes that for

1. \( F \) the forward rate, a random quantity,
2. \( \alpha \) the volatility of the forward rate, also a random quantity and,
3. \( X_1, X_2 \), two standard Brownian motions with correlation \( \rho \)
that

\[ dF = \alpha F^2 dX_1, \]

and

\[ d\alpha = v\alpha dX_2. \]

There are three parameters, namely, \( \alpha, \beta, \rho \). The advantage of this method is that there are good closed form approximations for the cases where the volatility \( \alpha \) and the volatility of the volatility \( v \) are both low. This is the case for bonds. Another advantage is that there are only three parameters to be estimated instead of the five of the Heston model.
6 Conclusion

In 1973 the Black Scholes Merton model was published. This made a huge difference to the pricing of options all over the financial world. As with any model it has to make assumptions about the market, these are (Gemmill, 1997, p14):

- the asset price is lognormally distributed, i.e. the returns on the asset are normally distributed,
- the distribution of the returns is constant,
- there are no transaction costs, so that hedging may be done without fees,
- the risk free interest rate is constant, and
- there are no dividends.

Quoting from Rebonato,

> Once we recognize that the assumptions of the Black-and-Scholes world are so strongly violated that we have to introduce a strike dependence on the implied volatility, the latter quantity simply becomes the wrong number to put in the wrong formula to get the right price of plain-vanilla options’ (Rebonato, 2004, p169, italics his).

It is clear that the CV model needs to be improved. The Heston model relaxes the first and second assumptions. We have shown in the Monte Carlo simulations that SV is more realistic than constant volatility models, we have also proposed that SV is not the end of the story and that the SVJ models might be an improvement. This is clearly a good direction for further study.
There are wider questions one can raise in this field. As one builds models of market behaviour what units of time should be used? Is it reasonable to assume that chronological time is sufficient? What about gaps between trading days, do we ignore these gaps? Would it be better to use the sequence of trades or the number of shares sold, as units of time? Can we assume that all trades are actually done for the purpose of trading or can there be other reasons, for example tax benefits? There are more questions in this field than one can hope to address.

Consider the path of a projectile near the surface of the earth. A first model of its trajectory is to ignore air resistance and assume gravity is constant and parallel, this gives us the familiar parabolic path. By the time we incorporate air resistance and the rotation of the earth we have some very messy mathematical equations but we have a more accurate model. This is the level we may be at with the improvement from CV to SV, SVJ and SABR models. We are certainly nowhere near the hermeneutic change from Newtonian physics to relativistic mechanics with it’s extreme accuracy. It might however be permissible to quote from Einstein,

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.’(J R Newman, 1956)

It is unlikely that economic phenomena will ever be as predictable as physical phenomena, not while trading decisions are made by human beings and their computer programmes. So, we are left chasing the gold at the end of the rainbow..., and occasionally finding some.
This section of the report contains the code used in the computer programmes which did the calculations producing the tables and graphs.

### A Numerical Integration

#### A.1 Matlab Code for Explicit Solution

The following Matlab code, D.m and F.m, was used to evaluate the functions $D$ (4.16) and $F$ (4.19).

```matlab
function {DJ} = D(u,j)

% We begin by entering the Heston parameters used here
lambda = 1.3253; rho = -0.7165; eta = 0.3877;

% This is a constant which we may vary at some stage
tau = 1/250;

% These are the intermediary functions
alpha = -u^2/2 -1i{u}/2 + 1i{u}{j}; beta = lambda-rho{eta}{j} - rho{eta}{1i{u}};
gg = eta^2/2; d = sqrt(beta^2 - 4{alpha}{gg});
rm = (beta - d)/eta^2; rp = (beta + d)/eta^2; g = rm/rp;

% We define D
DJ = rm{1-exp(-d{tau})}/(1-g{exp(-d{tau}))};
```

57
function \{FJ\} = F(u,j)

% We begin by entering the Heston parameters used here
lambda = 1.3253; rho = -0.7165; eta = 0.3877;

% This is a constant which we may vary at some stage
tau = 1/250;

% These are the intermediary functions
alpha = -u^2/2 - 1i*{u}/2 + 1i*{u}j; beta = lambda - rho*eta*j
- rho*eta*1i*u; gg = eta^2/2; d = sqrt(beta^2 - 4*alpha*gg);
rm = (beta - d)/eta^2; rp = (beta + d)/eta^2; g = rm/rp;

% We define F
FJ = lambda*( rm*tau - (2/eta^2)*log((1-g*exp(-d*tau))/(1-g))
);

Using the above code to calculate F and D, the integrals P_1 and P_0 of (4.18) were calculated using the following code, P.m.

function \{PJ\} = P(j,x)

% We enter the remaining Heston parameters

58
vbar = 0.0354; v = 0.0174;

% We evaluate the integral in 2.13 of p19 of Gatheral, 2006
gap = 0.001; PJI = 0;

for z=1:fix(1000/gap) PJI = PJI + real((exp(F(z*gap,j)*vbar + D(z/1000,j)*v + 1i*(z*gap)*x))/(1i*(z*gap))); end PJ = 1/2 + (1/pi)*PJI*(gap);

Finally the price of a call option was calculated using C.m. To operate this just enter the strike price, K.

function [CJ] = C(K)

% From eqn 2.5 on p17, Gatheral, 2006
x = log(1227.73/K); CJ = 1227.73*x*P(1,x) - K*x*P(0,x);

A.2 Programmes Supporting Reliability of the Explicit Solution

Q.m is the integrand of (4.18) which is taken from zero to infinity. The aim of this is to show that 1,000 is large enough to give a reliable upper limit for the numerical integral.
function \{[]\} PJI\{\} = Q(j,x,u)

% We enter the remaining Heston parameters for 15th Sept 2005
vbar =0.0354; v =0.0174;

PJI = real((exp(F(u,j){*}vbar + D(u,j){*}v + 1i{*}(u){*}x))/(1i{*}(u)));

PlotofM.m shows that it is reasonable to calculate (4.18) with an upper limit of 1,000.

function{[]} K{[]} = PlotofM(K)

x=log(1227.73/K); M=0;N=0; % R is the resolution of the plot, i.e. the number of points
R=50; for b=1:R N(b)=b{*}10; M(b)=K{*}(exp(x){*}Q(1,x,10{*}b)-Q(0,x,10{*}b));
end plot(N,M) StrikePrice = K;

**B Monte Carlo Milstein Discretization**

Table 4.2 was obtained using the following Matlab code on an Intel dual core using Matlab Version 7.2.0232 (R2006a). The second version referred to in the text was made using an Intel Celeron 1.6GHz, Matlab Version 7.7.0 (R2008b) and
used more iterations, \( N = 20,000 \) instead of \( N = 5,000 \), and a smaller time gap, 
\( \Delta t = 10^{-6} \) instead of \( \Delta t = 10^{-5} \). It produced almost the same results, but took
about six time as long to run.

Milstein.m is the Matlab programme which used the Monte Carlo method to
calculate asset prices and hence option prices using the Milstein Discretization
from Gatheral, 2006, p22.

% The results from this one seem quite reasonable. using \( N=5000 \) and
% \( \Delta t=0.00001 \).
% We use the initial price 'So' note use of lower case letter as it
% looks better.
% We get the final price and the price of a plain vanilla.
% Milstein Discretization of the Heston Model page 22 of Gatheral with
% parameters from p40, ie the 15th of Sept 2005.

function \{[]\}K\}= Milstein(K,eta,lambda,vbar,dt,T,So,rho,v0)

if nargin < 1 K = 1160 ; end; if nargin < 2 eta = 0.3877; end; if
nargin < 3 lambda = 1.3253; end; if nargin < 4 vbar = 0.0354; end;
if nargin < 5 dt = 10^-6; end; if nargin < 6 T = 1/250; end; if nargin
< 7 So = 1227.73; end; if nargin < 8 rho = -0.7165; end; if nargin
< 9 v0 = 0.0174;
end; \% x = \log( S_t / S_0 ), it's not the stock price, so it can be negative

for N=1:5000

    x0 = 0; v = v0; x = x0; xx=0; vv=0; SToverSo = 0;

    for j=1:(floor(T/dt)); R1 = randn; R2 = randn; v = abs((sqrt(v) + (eta/2) {*} sqrt(dt) {*} R1)2 - lambda {*} ( v - vbar) {*} dt - (eta2)/4 {*} dt); x = x + sqrt(v {*} dt) {*} ( rho {*}R1 + sqrt(1-rho2) {*} R2 ); \% xi = log of Si/S0. We have used the formula for a bivariate \% normal distribution from Glasserman, 2004, p72.
    vv(j) = v; xx(j) = x;

    end

    \% Here we calculate ST/So

    SToverSo(N) = K{*}exp(x); Result(N) = x; \%this is a plot like fig 4.1 on p.45 of Gathera

\%Here we calculate the price of plain vanillas
Call(N) = max(So{*}exp(x) - K,0); Put(N) = max(K - So{*}exp(x),0);

dend

% We may display the output with the command
hist(Result,40);

MeanEndpriceST = mean(SToverSo);

% Calculating the price of the options and 95% confidence intervals

MeanofCall = mean(Call) ConIntCall = 1.96{*}std(Call)/sqrt(N)

MeanofPut = mean(Put) ConIntPut = 1.96{*}std(Put)/sqrt(N)
References


